

ON PATH INTEGRAL LOCALIZATION AND THE LAPLACIAN, THE THESIS

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In this thesis, we develop path integral localization methods that are familiar from topological field theory: the integral over the infinite dimensional integration domain depends only on local data around some finite dimensional subdomain.

We introduce a new localization principle that unifies BRST localization, the non-Abelian localization principle and the conformal generalization of the Duistermaat-Heckman integration formula.

In addition, it is studied if one can possibly derive a generalized Selberg's trace formula on locally homogeneous manifolds using localization techniques. However, a definite answer is obtained only in the Lie group case (we complete the work of R. Picken) in which it is an application of the Duistermaat-Heckman integration formula. Also a new derivation of DeWitt's term is reported.

Furthermore, connections between evolution operators of integrable models and localization methods are studied. A derivative expansion localization is presented and it is conjectured to apply also to integrable models, for example the Toda lattice.

Moreover, a pedagogical introduction to the localization techniques is given, as well as a list of selected references that might be useful for a beginning graduate student in mathematical physics or for a mathematician who would like to study the physical point of view to topological field theory and string theory.

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- I. T. Kärki and A. Niemi, On the Duistermaat-Heckman integration formula and integrable models, Ahrenshoop Symp. 1993, 175; hep-th/9402041.
- II. T. Kärki and A. Niemi, *Supersymmetric quantum mechanics and the De Witt effective action*, Phys. Rev. D. **56**, 2080 (1997).
- III. T. Kärki, *On path integral localization and the Laplacian*, J. Math. Phys. **40**, 1807 (1999); hep-th/9712169.

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Chapter 1

Introduction

Ever since I. Newton invented differential calculus simultaneously with G. Leibniz the development of theoretical physics and mathematics have been closely related.

In the recent decades this interplay has manifested, for instance, in integrable models, topological field theories and string theory. Such abstract mathematics as Donaldson theory, Floer cohomology, Jones polynomial, Calabi-Yau manifolds, representation theory of infinite dimensional Lie groups, intertwining operators, modular forms, non-commutative geometry, quantum groups etc. have become familiar to physicists. This is hardly the end, the spectacular development of the second string revolution has given hope that all the five string theories can be unified in a single M theory in 11 dimensions (not to mention D-branes and black hole physics). When it is achieved, it is probably going to effect mathematics in a profound way as well as our understanding of space-time¹, of which we have had some flavour already in the form of non-commutative geometry. For the interested reader we have collected in Appendix B some references that we have found useful for further reading.

A very central object in todays mathematical physics is the partition function that is formulated as Feynman's path integral. Developing methods to calculate it is of great importance because it may enrich our understanding of non-perturbative quantum field theories. A lot is already understood: topological field theories, conformal field theories, integrable models, supersymmetric field theories and various dualities provide exact results and intriguing conjectures.

¹We thank C. Montonen for pointing this out.

The research in this thesis has concentrated on one particular phenomenon: path integral localization. It is familiar from topological field theories, the integral over the infinite dimensional integration domain actually depends only on local data around some finite dimensional subdomain. More specifically, we have been interested in if this method has applications beyond the apparently ‘trivial’ topological field theories. As a testing ground we have chosen the most trivial case, namely the Laplacian on locally homogeneous manifolds, that should not be overlooked as it often happens that mathematical structures repeat themselves in the most unexpected places. It is reasonable that understanding of this specific model might help one to understand also gauged WZW models that involve the most obvious generalization of Lie groups, the affine groups (or perhaps alternatively σ -models, our recent unpublished research suggests however rather WZW models). Not to mention that developing the localization method is important in its own right, we conjecture that in the long run it might increase our understanding of evolution operators of integrable models.

This thesis is organized as follows: In Ch. 2 we give an elementary introduction to the BRST symmetry which is the basis for localization. In Ch. 3 we illustrate the localization technique with the general Laplacian that serves also as a toy model of topological field theory. In Ch. 4 we introduce the model of our main interest, the Laplacian on functions, and comment on the semiclassical exactness of it on compact Lie groups, or in other words on the Duistermaat-Heckman integration formula in the infinite dimensional setting.

Then we advise the reader to papers I-III hoping that the preceding chapters have provided a pedagogical introduction. The goal of the research has been to derive generalized Selberg’s trace formulae on locally homogeneous manifolds using physicist’s path integral. They are of interest for both mathematicians and physicists because they can be used in calculating high loop string amplitudes that result in determinants of the Laplacian on Riemann surfaces. The goal is not achieved but some other interesting results are obtained: A new derivation of DeWitt’s term, a generalized canonical transformation that unifies previously unrelated localizations, new formulas on homogeneous spaces that might be useful in a hypothetical Selberg’s trace formula localization, a non-trivial localization deformation on homogeneous spaces, a derivative expansion localization that might also be useful for integrable models and a contribution to the problem ‘Why is the semiclassical approximation exact for the heat kernel on Lie groups?’. The last contribution is clarified in Ch. 4 of this thesis.

Chapter 2

BRST symmetry in gauge theories

The BRST symmetry is the basis for path integral localization. We review¹ it shortly in the historical context where it was discovered, in Yang-Mills theories: We begin with the Faddeev-Popov procedure. The partition function for four dimensional Yang-Mills can be written as

$$Z = \int [dA] \delta(\mathcal{G}^A) \det \left| \frac{\delta \mathcal{G}}{\delta \omega} \right| \exp S_c. \quad (2.1)$$

\mathcal{G}^A is the gauge condition, eg. $\mathcal{G}^B = \partial_\mu A^{B\mu}$ is the Lorentz gauge, and

$$S_c = \int d^4x \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \quad (2.2)$$

is the Yang-Mills action [1] for some gauge group G . The differentiation $\frac{\delta \mathcal{G}}{\delta \omega}$ requires some further explanation, the variation in the numerator actually means the effect of an infinitesimal gauge variation on the gauge condition \mathcal{G} that is parametrized by ω^B . More explicitly, $\delta A_\mu^B = D_\mu \omega^B$ so that for the Lorentz gauge $\frac{\delta \mathcal{G}}{\delta \omega} = \partial_\mu D^\mu$, where $D_\mu = \partial_\mu + A_\mu$ is the covariant derivative.

The Faddeev-Popov procedure introduces a functional Fourier transform of the delta function²

$$\delta(\mathcal{G}) = \int [db] \exp \int b^A \mathcal{G}^A \quad (2.3)$$

¹Our exposition follows closely L. Baulieu's unpublished lecture notes.

²Compare $\delta(x) \sim \int_{-\infty}^{\infty} e^{ipx} dp$.

and a ghost path integral representation of the determinant

$$\det \left| \frac{\delta \mathcal{G}}{\delta \omega} \right| = \int [dc d\bar{c}] \exp \int \bar{c} \frac{\delta \mathcal{G}}{\delta \omega} c \quad (2.4)$$

where c^B, \bar{c}^B are Grassmann fields. In the Lorentz gauge the resulting path integral is

$$Z = \int [dA db dc d\bar{c}] \exp \int \text{tr} F_{\mu\nu}^2 + b\partial \cdot A + \bar{c}(\partial \cdot D)c, \quad (2.5)$$

we denote the action in (2.5) as S_q in contrast to the action in (2.2). It possesses a BRST symmetry Q , $QS_q = 0$, which is a supersymmetry (it exchanges bosonic and fermionic fields) and satisfies $Q^2 = 0$. Its action on the fields is

$$QA_\mu^B = D_\mu c^B \quad (2.6)$$

$$Qc^A = -\frac{1}{2}f^{ABC}c^Bc^C \quad (2.7)$$

$$Q\bar{c}^A = b^A \quad (2.8)$$

$$Qb^A = 0 \quad (2.9)$$

and it is a graded derivation (ie. like the exterior derivative on forms) by the grading of the ghost number, b and A are even while the ghosts c, \bar{c} are odd.

S_q can be written as the Yang-Mills action S_c plus a BRST exact term

$$S_q = S_c + Q\psi \quad (2.10)$$

where

$$\psi = \bar{c}^B \partial \cdot A^B. \quad (2.11)$$

The Batalin-Vilkovisky theorem states that the partition function (2.5) is independent of the gauge fermion ψ , different ψ 's correspond to different gauge fixings. It can be proved as follows [2]: Consider the partition function

$$Z = \int [d\Phi] \exp S[\Phi] + Q\psi \quad (2.12)$$

where Φ means collectively all the fields (A, b, c, \bar{c}). An infinitesimal change of variables ($\delta\psi$ is an infinitesimal gauge fermion ie. infinitesimal real variable times a gauge fermion)

$$\delta\Phi = \delta\psi Q\Phi \quad (2.13)$$

gives a super-Jacobian $\exp Q\delta\psi$ and the action does not change because of the BRST symmetry $QS = 0$. Thus, ψ in equation (2.12) is replaced by $\psi + \delta\psi$ and therefore the partition function (2.12) is independent of ψ . \square

For example, if one considers instead the gauge fermion

$$\psi_{\text{standard}} = \bar{c}^B (\mathcal{G}^B + \alpha b^B), \quad (2.14)$$

one gets after the integration of the auxiliary b field

$$S = \int \text{tr} F_{\mu\nu}^2 + \frac{1}{2\alpha} \mathcal{G}^A \mathcal{G}^A + \bar{c} \left(\frac{\delta \mathcal{G}}{\delta \omega} \right) c$$

which is one of the standard gauge fixings.

Chapter 3

Localization, the Laplacian and index theorems

We illustrate path integral localization with the Laplacian, some excellent reviews are Refs. [3] and [4]. Despite the apparent triviality, this model shows many of the features of topological field theories (Donaldson theory, topological σ -models, BF-theories etc.).

On a Riemannian manifold one can write the general Laplacian on forms as

$$\Delta = dd^* + d^*d = [d, d^*]_{\text{graded}}. \quad (3.1)$$

The Z_2 grading of the exterior algebra is

$$\bigwedge M = \left(\bigoplus_{k=0,2,\dots} \bigwedge^k M \right) \oplus \left(\bigoplus_{k=1,3,\dots} \bigwedge^k M \right) = \wedge_+ M \oplus \wedge_- M,$$

and it induces also a grading for the linear operators that map $\bigwedge M$ onto itself. The exterior derivative d and its dual d^* are odd (or fermionic) operators because they map even forms to odd forms and vice versa.

The partition function in operator formalism is

$$Z = \text{Str} e^{-\beta[d, d^*]}, \quad (3.2)$$

where $\text{Str} = \text{tr}(-1)^N$ and N is the operator that gives the degree of the form, eg. for a p-form ω , $N\omega = p\omega$.

The path integral presentation of the partition function (3.2) is

$$Z = \int [\sqrt{g} dx^\mu d\psi^\mu d\bar{\psi}_\mu] \exp \int_0^\beta g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \bar{\psi}_\mu \nabla_t \psi^\mu + R_{\kappa\lambda}^{\mu\nu} \bar{\psi}_\mu \bar{\psi}_\nu \psi^\kappa \psi^\lambda \quad (3.3)$$

where one has periodic boundary conditions for both the bosons x^μ and the fermions $\bar{\psi}_\mu$, ψ^μ . The action in (3.3) results from canonical quantization [5],

$$S = \int p_\mu \dot{x}^\mu + \bar{\psi}_\mu \dot{\psi}^\mu - \{d, d^*\} \quad (3.4)$$

which after integration of the auxiliary field p gives exactly (3.3). In equation (3.4) d , d^* are the classical limits of the corresponding operators, ie. functions on the super phase space, and $\{ , \}$ is the graded Poisson bracket

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu \quad (3.5)$$

$$\{\psi^\mu, \bar{\psi}_\nu\} = \delta_\nu^\mu \quad (3.6)$$

that is associated with the kinetic term $\int p_\mu \dot{x}^\mu + \bar{\psi}_\mu \dot{\psi}^\mu$. In the following we use also the loop space version of it which is obtained as follows: if

$$\omega = \omega_{\mu\nu} d\phi^\mu \wedge d\phi^\nu$$

is the corresponding symplectic form on the phase space Γ with local coordinates ϕ^μ , then the loop space symplectic form is

$$\omega = \int_0^\beta \omega_{\mu\nu} \delta\phi^\mu \wedge \delta\phi^\nu.$$

The action in (3.4) can be written as a BRST exact variation

$$S = Q \left(\int \dot{x}^\mu \bar{\psi}_\mu - d^* \right)$$

where the BRST operator $Q = \{d, \cdot\}$ satisfies

$$Qx^\mu = \{d, x^\mu\} = -\psi^\mu \quad (3.7)$$

$$Qp_\mu = 0 \quad (3.8)$$

$$Q\psi^\mu = 0 \quad (3.9)$$

$$Q\bar{\psi}_\mu = p_\mu. \quad (3.10)$$

One can argue that because the BRST operator does not depend on the metric and because of the Batalin-Vilkovisky theorem, the partition function is independent of the metric. Thus, it is a topological invariant. In the Hamiltonian formalism it is even more evident: The Hamiltonian operator $\hat{H} = \Delta$ satisfies

$$[d, \hat{H}] = 0$$

so that we can choose the exterior derivative d to be the BRST operator Q . The Batalin-Vilkovisky theorem in the operator formalism can be stated and proved as follows:

Theorem 1 *Provided that the Hamiltonian \hat{H} has a BRST symmetry Q , ie. fermionic operator that satisfies*

$$[Q, \hat{H}] = 0, \quad [Q, Q] = 0,$$

the partition function

$$Z = \text{Str} e^{-\beta \hat{H} + [Q, \psi]} \quad (3.11)$$

is independent of the odd operator (gauge fermion) ψ .

Proof. We define

$$Z_\lambda = \text{Str} e^{-\beta \hat{H} + [Q, \psi_\lambda]},$$

where ψ_λ is a one-parameter family of gauge fermions. Then

$$\frac{\partial Z_\lambda}{\partial \lambda} = \text{Str} [Q, \frac{\partial \psi_\lambda}{\partial \lambda}] e^{-\beta \hat{H} + [Q, \psi_\lambda]} \quad (3.12)$$

$$= \text{Str} [Q, \frac{\partial \psi_\lambda}{\partial \lambda} e^{-\beta \hat{H} + [Q, \psi_\lambda]}] \quad (3.13)$$

which vanishes because the supertrace of a graded commutator is zero. It is easy to see that any two gauge fermions can be connected by a one-parameter family of such fermions. \square

In particular we see that the partition function (3.2), or equivalently (3.3), is independent of β because it can be absorbed in the gauge fermion. In the limit $\beta \rightarrow \infty$ it gives

$$Z = \sum_{i=0}^D (-1)^i \dim \ker \Delta_i = \chi(M)$$

where Δ_i is the Laplacian on i -forms. On the other hand, evaluating the partition function in the limit $\beta \rightarrow 0$ gives the heat kernel proof of the Gauss-Bonet index theorem [6]. The result is an integral over the Euler class of the tangent bundle,

$$Z = \int_M \text{Pf } R, \quad (3.14)$$

where R is the curvature tensor. It can be seen most easily in the path integral form (3.3). The limit $\beta \rightarrow 0$ corresponds to the dimensional reduction: all the

fields $(x^\mu, \psi^\mu, \bar{\psi}_\mu)$ are considered to be independent of time, which results in a zero dimensional theory on the zero modes:

$$Z = \beta^{-D/2} \int \sqrt{g} dx^\mu d\psi^\mu d\bar{\psi}_\mu e^{\beta R_{\kappa\lambda}^{\mu\nu} \bar{\psi}_\mu \bar{\psi}_\nu \psi^\kappa \psi^\lambda} \quad (3.15)$$

$$= \int_M \text{Pf } R. \quad (3.16)$$

But, as we proved in Theorem 1, the operator d^* can actually be replaced by any odd operator ψ , which shows that one has a lot more freedom in evaluating the partition function. We consider another way of evaluating it which results in the Poincaré-Hopf theorem (or more generally the Matthes-Quillen theorem). We use E. Witten's Morse theoretic twist [7]

$$d \rightarrow d_s = e^{sf} de^{-sf}, \quad d^* \rightarrow d_s^* = e^{-sf} d^* e^{sf}, \quad (3.17)$$

(where it is assumed that f has only isolated critical points and that the Hessian is nondegenerate) and consider the generalized partition function

$$Z = \text{Str } e^{-\beta[d_s, d_s^*]}. \quad (3.18)$$

It is independent of s and β because it can be written as

$$\text{tr } e^{sf} \left((-1)^N e^{-\beta[d, d_{2s}^*]} \right) e^{-sf} = \text{Str } e^{-\beta[d, d_{2s}^*]}$$

and because of Theorem 1. When s is zero it coincides with the partition function (3.2). In the path integral form

$$Z = \int [\sqrt{g} dx^\mu d\psi^\mu d\bar{\psi}_\mu] \exp - \int_0^\beta g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \bar{\psi}_\mu \nabla_t \psi^\mu + R_{\kappa\lambda}^{\mu\nu} \bar{\psi}_\mu \bar{\psi}_\nu \psi^\kappa \psi^\lambda \quad (3.19)$$

$$+ s \bar{\psi}_\mu g^{\mu\kappa} \frac{D^2 f}{Dx^\kappa Dx^\nu} \psi^\nu + s^2 g^{\mu\nu} \partial_\mu f \partial_\nu f$$

which in the limit $\beta \rightarrow 0$ gives using the dimensional reduction (see Appendix A for more details)

$$Z = \beta^{-D/2} \int \sqrt{g} dx d\psi d\bar{\psi} \exp -\beta(R_{\kappa\lambda}^{\mu\nu} \bar{\psi}_\mu \bar{\psi}_\nu \psi^\kappa \psi^\lambda \quad (3.20)$$

$$+ s \bar{\psi}_\mu g^{\mu\kappa} \frac{D^2 f}{Dx^\kappa Dx^\nu} \psi^\nu + s^2 g^{\mu\nu} \partial_\mu f \partial_\nu f)$$

which is called the Matthes-Quillen form. In the limit $s \rightarrow \infty$ the integral localizes around the critical points $df = 0$. Thus, it is legitimate to expand in Taylor

series around the critical points and sum the contributions (it amounts to using the delta function formula $\delta(x) \sim \sqrt{\lambda} \exp -\lambda x^2$). For example $g^{\mu\nu} \partial_\mu f \partial_\nu f = g^{\mu\nu} \partial_{\mu\kappa} f \partial_{\nu\lambda} f x^\kappa x^\lambda$ where the coordinate system is chosen so that $x = 0$ at the particular critical point whose contribution is calculated.

The resulting integrals are Gaussian in the limit and can be summed to give

$$Z = \sum_{df=0} \text{sign det} \left(\frac{\partial^2 f}{\partial x^\mu \partial x^\nu} \right) = \sum_{df(p)=0} (-1)^{n_p}, \quad (3.21)$$

which is the Poincare-Hopf theorem. The integer n_p denotes the Morse index at point p , ie. the number of negative eigenvalues of the Hessian.

Concluding, we have showed that using mixed path integral and operator techniques it is relatively easy to derive Gauss-Bonnet and Poincare-Hopf index theorems. The methods that we have described can be used to derive many if not all the index theorems, see Refs. [8] and [6]. This is even more interesting because they can be understood as toy models of topological quantum field theory (see the references in Appendix B). For example, similarly some correlation functions of Donaldson theory, which is a twisted $N = 2$ supersymmetric Yang-Mills theory, can be localized on the moduli space of instantons giving Donaldson polynomials. Finally, for the reader who is interested in the Laplacian we mention that the partition function of the BF theories is essentially the Ray-Singer torsion [3]

$$T_M = \prod_{k=0}^D (\det \Delta_k)^{(-1)^k k/2}$$

(where Δ_k is the Laplacian on k -forms) that is a topological invariant.

Chapter 4

DH theorem and the Laplacian on Lie groups

The model of main interest in this thesis is the Laplacian on zero-forms Δ_0 . The partition function of it is in the path integral form on a general Riemannian manifold¹

$$Z = \text{tr } e^{-\beta \Delta_0} = \int [\sqrt{g} dx] e^{\int_0^\beta g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}. \quad (4.1)$$

The model is particularly interesting because it is known to possess localization behaviour: it is semiclassically exact on Lie groups and on Riemann surfaces it can be evaluated using Selberg's trace formula (see paper III). We comment on the former clarifying our result in paper III and completing the seminal work of R. F. Picken in Ref. [9].

On a compact Lie group G the heat kernel can be written as a path integral over the paths in G satisfying the boundary conditions $g(0) = \mathbf{1}$, $g(\beta) = g$:

$$k_\beta(\mathbf{1}, g) = \int [dg] \exp \int \text{tr}(g^{-1} \partial_t g)^2 \quad (4.2)$$

$$= \int [\sqrt{\det(K_{ij} \omega_\mu^i \omega_\nu^j)} dx^\mu] \exp \int K_{ij} \omega^i(\dot{x}) \omega^j(\dot{x}) \quad (4.3)$$

where

$$T_i \omega_\mu^i dx^\mu = g^{-1} dg$$

¹We neglect DeWitt's term (paper II), it can be thought of as being part of the measure.

are the left invariant vector fields and T_i are the generators of the Lie algebra having the Killing form

$$K_{ij} = -\text{tr } T_i T_j = - \langle T_i, T_j \rangle.$$

On the Lie group one has the unique bi-invariant metric

$$g_{\mu\nu} = K_{ij} \omega_\mu^i \omega_\nu^j.$$

The integral (4.3) can be transformed into an integral over the based loop space [9] using the change of variables

$$g(t) \rightarrow g_0(t)g(t) \quad (4.4)$$

where $g_0(t)$ is a geodesic connecting $\mathbf{1}$ and g ,

$$g_0(t) = e^{(J^i T_i) \frac{t}{\beta}} \quad (4.5)$$

where J^i are constants. The boundary conditions in the integral (4.3) change to periodic and the action to

$$S = \int_0^\beta K_{ij}(\omega^i(\dot{x}) + J^i)(\omega^j(\dot{x}) + J^j). \quad (4.6)$$

The loop space can be interpreted as an infinite dimensional flag manifold which in particular means that it is a Kähler manifold. From the localization point of view it is important that it possesses a right invariant symplectic form which at the unity is given by the Maurer-Cartan cocycle [9]

$$\omega(X, Y) = \int_0^\beta K_{ij} X^i \partial_t Y^j. \quad (4.7)$$

$X = X^i(t)T_i \in Lg$, $X^i(0) = 0$, is an element of the Lie algebra of the based loop space.

Using right translations one obtains the symplectic form

$$\omega = \int_0^\beta K_{ij} \omega_R^i \partial_t \omega_R^j \quad (4.8)$$

where $dg g^{-1} = T_i \omega_R^i$ are the right invariant one-forms. This can be seen easily by noticing that it is annihilated by the Lie derivatives $\mathcal{L}_{f^* J^i(t)v_i}$ (v_i are the left invariant vector fields that generate the right action) and that it coincides at the unit element with (4.7).

The Liouville measure associated with this symplectic form equals the natural bi-invariant measure $[\sqrt{g} dx]$ in equation (4.3), as has been argued heuristically in paper III, so that one can finally write

$$k_\beta(\mathbf{1}, g) = \int [\sqrt{\omega} dx] \exp \int K_{ij}(\omega^i(\dot{x}) + J^i)(\omega^j(\dot{x}) + J^j) \quad (4.9)$$

where the integral is over the based loop space. It has been shown in Ref. [9] that an application of the Duistermaat-Heckman integration formula

$$Z = \sum_{\delta S=0} \frac{\sqrt{\det \omega_{\mu\nu}}}{\sqrt{\frac{\delta^2 S}{\delta x^\mu \delta x^\nu}}} e^S \quad (4.10)$$

yields the exact expression for the heat kernel, which, as the measure coincides with the metric one, amounts to the semiclassical exactness of the theory.

We comment briefly on the path integral level of rigour why the Duistermaat-Heckman theorem can be applied. The point is that the Hamiltonian (4.6) generates a torus action² (probably it is even a projective Hamiltonian on the flag manifold in the sense of Ref. [9]) which allows one to use an invariant metric and the BRST localization principle.

We denote

$$H = \int_0^\beta K_{ij} \omega^i(\dot{x}) \omega^j(\dot{x}) \quad (4.11)$$

$$I_i = \int_0^\beta K_{ij} \omega^j(\dot{x}). \quad (4.12)$$

The corresponding Hamiltonian vector fields are

$$\tilde{\chi} = \dot{x} - \omega^i(\dot{x})|_{t=0} v_i^R \quad (4.13)$$

$$u_i = v_i - v_i^R. \quad (4.14)$$

The Poisson brackets of the Hamiltonians read

$$\{H, I_i\} = 0 \quad (4.15)$$

$$\{I_i, I_j\} = C_{ij}^k I_k. \quad (4.16)$$

The flow associated with $\tilde{\chi}$ is just the rotation of the loops

$$g(t) \rightarrow g(\alpha)^{-1} g(t + \alpha) \quad (4.17)$$

²We thank O. Tirkkonen for conjecturing this and the following comment in the parenthesis.

and the vector field u_i generates the adjoint action

$$g(t) \rightarrow h(\gamma)g(t)h(\gamma)^{-1} \quad (4.18)$$

where $h(\gamma) = e^{\gamma T_i}$ [10].

Altogether, we have a Hamiltonian action of the group $U(1) \times G$ on the phase space of based loops ΩG . We suppose in addition that $J^i T_i$ is in the generic position that the elements of the Lie algebra that annihilate it generate a Cartan subalgebra H [11], we denote by T the associated maximal torus. Then the torus $U(1) \times T$ has a Hamiltonian action on the phase space and in particular the Hamiltonian

$$S = \int K_{ij}(\omega^i(\dot{x}) + J^i)(\omega^j(\dot{x}) + J^j) = H + J^i I_i + \text{constant} \quad (4.19)$$

generates an element of the torus group. We construct an invariant metric g by averaging an arbitrary metric \tilde{g} over the action of the torus³

$$g = \int_{\vec{t} \in U(1) \times T} d\vec{t} \varphi_{\vec{t}}^* \tilde{g} \quad (4.20)$$

where $\varphi_{\vec{t}}$ is the diffeomorphism associated with the element \vec{t} of the torus. Then the Hamiltonian vector field χ associated to the action (4.19) satisfies the Lie derivative condition

$$\mathcal{L}_\chi g = 0 \quad (4.21)$$

and we are in position to apply the BRST proof of the Duistermaat-Heckman theorem (paper I).

We exponentiate the symplectic measure by introducing a Grassmann integral

$$k_\beta(\mathbf{1}, g) = \int [dx^\mu d\psi^\mu] e^{S+\omega} \quad (4.22)$$

and using the BRST symmetry

$$(d + i_\chi)(S + \omega) = 0, \quad (4.23)$$

where i_χ is the contraction operator by the vector field χ , we add the gauge fermion

$$\psi = \lambda i_\chi g$$

³It is enough to average over the group $U(1) \times G$ because G is compact, however, we would like to emphasize the appearance of a torus action.

obtaining

$$k_\beta(\mathbf{1}, g) = \int [dx^\mu d\psi^\mu] \exp(S + \omega - \lambda g(\chi, \chi) - \lambda d(i_\chi g))$$

which in the limit $\lambda \rightarrow \infty$ localizes on the zeros of χ giving the Duistermaat-Heckman formula (4.10).

In paper III we have constructed explicitly an invariant tensor g' which localizes equally well, if it would be nondegenerate the treatment there would be exactly parallel to the one given here. And finally, we end the discussion by mentioning that the semiclassical exactness holds also on the noncompact groups [12], but it seems to be difficult to explain it using similar path integral arguments.

Summary of the papers

- I.** The integral proof of the Duistermaat-Heckman theorem is reviewed. Connections between the Duistermaat-Heckman theorem and integrable models are speculated. A geodesic condition for the Hamiltonian vector field is reported as a possible generalization of the isometry condition appearing in the DH theorem. A bi-Hamiltonian structure associated with the new condition is presented and it is speculated if it underlies a new localization principle.
- II.** A new derivation of DeWitt's term is presented. It results from considering the grand canonical partition function of the supersymmetric quantum mechanics associated with the Laplacian on general forms. The Laplacian on zero-forms together with DeWitt's term appears in the limit that the chemical potential is put to infinity. The numerical value of DeWitt's term seems to be connected with the cancellation of quantum mechanical anomalies.
- III.** A new localization principle is reported. It unifies BRST localization, non-Abelian localization principle and the conformal DH formula of Paniak, Semenoff and Szabo. It is also applied to the Laplacian on homogeneous manifolds but it fails to give the speculated localization that would generalize Selberg's trace formula. A derivative expansion is used to localize the Laplacian on Lie groups and it is conjectured to apply also to integrable models. In addition the semiclassical exactness on Lie groups is explained by completing R. F. Picken's work using a path integral proof.

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Appendix A

Technical details

We explain in a little more detail the dimensional reduction:

We consider the partition function

$$Z = \int [dx dp d\psi d\bar{\psi}] \exp \int_0^\beta p_\mu \dot{x}^\mu + \bar{\psi}_\mu \dot{\psi}^\mu - H_s \quad (\text{A.1})$$

$$= \int [dx dp d\psi d\bar{\psi}] \exp \int_0^1 p_\mu \dot{x}^\mu + \bar{\psi}_\mu \dot{\psi}^\mu - \beta H_s, \quad (\text{A.2})$$

where $H_s = \{d_s, d_s^*\}$, and use periodic boundary conditions for all the fields. The limit $\beta \rightarrow 0$ can be evaluated by scaling time dependent modes x_t ($x^\mu = x_0^\mu + x_t^\mu$, where x_0 is the constant Fourier mode and x_t the rest of the expansion). More explicitly, we use the change of variables

$$x_t \rightarrow \sqrt{\beta} x_t, \quad p_t \rightarrow \sqrt{\beta} p_t \quad (\text{A.3})$$

$$\psi_t \rightarrow \sqrt{\beta} \psi_t, \quad \bar{\psi}_t \rightarrow \sqrt{\beta} \bar{\psi}_t \quad (\text{A.4})$$

and because the measure splits as

$$dx_0 dp_0 d\psi_0 d\bar{\psi}_0 [dx_t dp_t d\psi_t d\bar{\psi}_t]$$

the Jacobian is 1. The action becomes

$$S = \beta \int_0^1 p_{t\mu} \dot{x}_t^\mu + \bar{\psi}_{t\mu} \dot{\psi}_t^\mu - H(x_0, p_0, \psi_0, \bar{\psi}_0) + O(\beta^{1/2}) \quad (\text{A.5})$$

and in the limit $\beta \rightarrow 0$ one can evaluate the integral over the time dependent modes

$$\int dx_t d\psi_t \delta(\dot{x}_t) \delta(\dot{\psi}_t) = \sqrt{\frac{\det \delta_\nu^\mu \partial_t}{\det \delta_\nu^\mu \partial_t}} = 1 \quad (\text{A.6})$$

so that the final result is

$$\int dx_0 dp_0 d\psi_0 d\bar{\psi}_0 \exp -\beta H_s \quad (\text{A.7})$$

where β is small (actually the result is equivalent to the classical limit $\hbar \rightarrow 0$ which is not so surprising because β and \hbar appear in almost the same way). However, because of the BRST structure $H_s = \{d_s, d_s^*\}$ the result is independent of β so that it can be considered to be finite instead. In Eqs. (3.15), (3.20) the factor $\sqrt{g}\beta^{-D/2}$ results from integrating the dp_0 integral.

Appendix B

Selected references

B.1 Field theory

Basic references on quantum field theory are [1], [13], [14], [15], [16] and [17]. In order to understand renormalization and for many other reasons it is good to have some idea of lattice field theory [18], critical phenomena in statistical physics and Wilson's renormalization group. For the two latter we unfortunately do not know any particularly good reference (except perhaps conformal field theory on which we comment later). Some information on solitons, monopoles and instantons can be found in Refs. [19], [20], [21], [22] and [23], for deeper understanding we suggest integrable models and duality in supersymmetric gauge theories on which we comment later. Some references on constrained quantization and BRST are Ch. 2 of this thesis and Refs. [24], [2], [25], [26]. Topological field theory provides perhaps the best place to see it in action, see the section on toy models below. Important background for grand unified theories and almost all theoretical physics are Lie groups, the best reference for physicists can be found in one chapter of the book [27]. One book on grand unified theories is [28] and a very short and easy introduction can be found in Ref. [29]. We also recommend to take a look at the end of the volume two of the superstring theory book [30]. The Fujikawa method for anomalies is described in Ref. [17], see also [14] and [30] for the Feynman graph point of view. But we recommend to study string theory for deeper understanding, see the references in string theory section below. For confinement see Refs. [31], [32] and the following section on supersymmetry.

However, the best way to learn field theory is to study string theory, which, despite the apparently naive idea, leads to a surprisingly realistic theory. Many recent advances in field theory have been derived from string theory in the low energy field theory limit.

B.2 Supersymmetry

Supersymmetry, although not observed in the nature, is of great importance in theoretical physics.

Good basic references are [15], [33], [34], [35], [36]. The fastest way to learn supersymmetry is to study the exact solution of $N = 2$ supersymmetric Yang-Mills theory that was discovered by Seiberg and Witten. One of the most pedagogical references is [37], but there are also a lot of other lecture notes on the internet [38] not to mention the original papers [39].

The importance of Seiberg-Witten theory is difficult to exaggerate [40]. It has proved that t'Hoofts monopole condensation mechanism [32] actually occurs in certain supersymmetric theories, and because Donaldson theory is a twisted $N = 2$ theory it has had in addition remarkable consequences for topology. Furthermore this duality has been extended to string theories leading to the discovery of M theory. We give some references on these other developments below.

B.3 String theory

And finally, we come to the most important section of this appendix: String theory. The most pedagogical references are perhaps the books [30], [15], [41] together with the unpublished proceedings [42] and the thesis [43]. For string theory compactifications (see also Ref. [44] where the original Kaluza-Klein compactification is presented: 5 dimensional gravity gives an effective theory of gravity and electrodynamics in the Minkowski space) one needs also background in algebraic geometry, for which we recommend Refs. [30], [45] and [46]. The discovery of D-branes and dualities have revolutionized the subject and despite Polchinski's book [41], and perhaps the proceedings [47] and the thesis [43], the information is scattered on the internet (hep-th bulletin board: xxx.lanl.gov; slac-spires conference and proceedings info: www.slac.stanford.edu/find/spires.html, especially Nordita and Trieste conferences; various internet sites which gather reviews

on the web; various institutes that have also online proceedings,...).

We give some references for these more recent developments, but apologize for possible misleading comments as we are not really specialists in string theory:

The basic idea of describing branes as strings with Dirichlet boundary conditions (ie. branes are objects on which the end points of open strings must move) is explained in [48]. A very pedagogical treatment can be found in Ref. [49]. In the low energy field theory they are brane-like solitons, see Refs. [50] and [51]. The string theory dualities are treated pedagogically in Ref. [52] but also E. Witten's original paper [53] is very pedagogical. But there are also a lot of other lectures on the subject on the internet, for example Refs. [54]. The string dualities lead also to the conjecture about M theory which in the low energy limit should be the eleven dimensional supergravity. It is the only supergravity in dimension 11 and there are no supergravities in higher dimensions (if we forget F theory on which we do not comment at all, although it may be very interesting and important). The matrix model conjecture associated with M theory is described pedagogically in Refs. [41] and [55].

The unexpected bonus of the recent developments has been the increased understanding of ordinary field theories. On p -branes one has the effective field theory which is the D=10, N=1 super Yang-Mills dimensionally reduced to $p+1$ dimensions, the scalars of the theory describe the fluctuations of the brane. The $SU(N)$ gauge theory in $p+1$ dimensions can be described as putting N D p -branes on the top of each other [56]. A very important recent development is the Maldacena conjecture which relates the large N structure of the $N=4$ theory with type IIB string theory [57]. It confirms the long speculated relation of large N gauge theory and string theory [13].

The most intriguing 'recent' triumph of string theory has been the black hole entropy calculation of C. Vafa using D-branes [58], it seems that string theory is finally approaching its original goal: understanding of quantum gravity.

From the mathematical point of view string theory is equally interesting: vertex operator algebras, modular forms, mirror symmetry (T-duality), Calabi-Yau manifolds, superconformal field theory and non-commutative geometry have found their place in physics. And there is probably a lot more to be discovered as the final formulation of M theory is still lacking.

B.4 Toy models

Some insights into field theory and string theory can be obtained by studying different toy models, which are also very important in their own right. A very quick introduction to many of the subjects in this section can be found in Ref. [59].

Conformal field theory in two dimensions is important because it is the mathematical structure underlying string theory and because it is solvable (minimal or rational CFTs) it provides many insights into quantum field theory. Particularly good references are [60], [61], [27] and [41]. The mathematicians point of view can be found in Ref. [62]. We recommend the reader to study in particular the cosets models that are the gauged WZW models (rational CFTs) and have the enlarged affine group symmetry.

Even more interesting is superconformal field theory because its relations to mirror symmetry, quantum cohomology, topological sigma models on Calabi-Yau manifolds etc. Unfortunately our knowledge ends here.

Integrable models are another source of intuition to quantum field theory, in particular to soliton physics, see Refs. [63], [64], [65], [66] and [67]. Integrable models are the integration tables of our time and they repeat themselves in all branches of physics, notably in topological field theories like two dimensional gravity. The field is so large that the reference list given here is very subjective.

Good references on topological field theory are [68], [3], [4] and [69]. See Ref. [70] for background on knots. We recommend the reader also to take a look at the papers of M. Blau and G. Thompson on the hep-th bulletin board. For Seiberg-Witten invariants we recommend the references in the previous supersymmetry section and to search the internet for lectures, eg. Refs. [71]. A particularly interesting model of topological field theory is Chern-Simons theory, which has also a connection to three dimensional gravity. One might even gain understanding of black holes by studying such toy models.

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